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Advantages of Quantum Mechanics on Phase Space

F. E. Schroeck, Jr.[1]

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Quantum mechanics formulated in terms of (wave) functions over phase space is shown to have numerous advantages over the standard approach. These advantages arise in the contexts of discussion of the theoretical framework and of descriptions of laboratory experiments.

1. THE STRUCTURE OF THE THEORY OF QUANTUM MECHANICS ON PHASE SPACE

Quantum mechanics on phase space arises in the following setting: Let $\mathfrak{S}$ and $\mathfrak{A}$ be sets. Elements of $\mathfrak{S}$ are termed "states" and elements of $\mathfrak{A}$ are termed "observables." G will denote a (Lie) group of automorphisms of $\mathfrak{A}$. H will denote any closed subgroup of G for which the homeogeneous space G/H of G possesses a symplectic structure. $\mathfrak{E}$ is a set of positive bilinear mappings: $\mathfrak{S} \times \mathfrak{A} \to \mathbb{R}$; elements of $\mathfrak{E}$ will be called (quantum) expectations. Then a (quantum mechanical) physical system is a quadruple $\{\mathfrak{S}, \mathfrak{A}, G, \mathfrak{E}\}$ satisfying the following:

Axiom 1. Every state in $\mathfrak{S}$ corresponds to a density operator ρ in $L^2_\mu(G/H)$ for some symplectic homogeneous space G/H of G, where μ is the invariant measure arising from the symplectic two-form. Conversely, to each such density operator there corresponds a state.

In particular, if $\psi, \varphi \in L^2_\mu(G/H)$, then $\psi + \beta\varphi \in L^2_\mu(G/H)$ for any $\beta \in \mathbb{C}$ and some state corresponds to this mixture. In this way, the superposition principle holds.

The modern approach to classical mechanics tells us that the property that G/H is a symplectic space is equivalent to G/H being identifiable as a classical phase space.

[1]Florida Atlantic University, Boca Raton, Florida.

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Axiom 2. The coordinate variables of the symplectic homogeneous space G/H have a prior interpretation as classical observables in classical phase space.

From Axiom 2, given Borel set Δ of classical phase space G/H, one may define and interpret the phase space localization operator $A(\Delta)$ on $L^2_\mu(G/H)$ given by

$$A(\Delta)\Psi = \chi_\Delta \Psi$$

for all $\Psi \in L^2_\mu(G/H)$, where χ_Δ is the characteristic function for set Δ. We remark that the map $A: \text{Borel}(G/H) \to \text{Projections on } L^2_\mu(G/H)$, $A: \Delta \mapsto A(\Delta)$, defines a projection valued measure (PVM).

Axiom 3. Elements of $\mathfrak{A}$ are realized in the setting of $L^2_\mu(G/H)$ as self-adjoint (or perhaps only symmetric) operators on $L^2_\mu(G/H)$. The map $A: \text{Borel}(G/H) \to \text{Projections on } L^2_\mu(G/H)$, $A: \Delta \mapsto A(\Delta)$, is such that the set of observables in $L^2_\mu(G/H)$ contains the span of $\{A(\Delta) | \Delta \in \text{Borel}(G/H)\}$ as well as the closure of the span (in a suitable topology which I will not describe here).

Since G/H is a homogeneous space of G, there is a natural action of G on this set of cosets of G. For $x \in G/H$, and $g \in G$, this action will be denoted $x \mapsto gx$. Let α denote any character ($\mathbb{C}$-valued representation) of H.

Define $\sigma: G/H \to G$ to be a Borel section and $\pi: G \to G/H$ to be the canonical projection. Then for $g \in G$ and $x \in G/H$, $h(g^{-1}, x) \equiv \sigma(x)^{-1} \circ g \circ \sigma(g^{-1}x)$ is an element of H. It follows that V^α, defined on $L^2_\mu(G/H)$ by

$$[V^\alpha(g)\Psi](x) \equiv \alpha(h(g^{-1}, x))\Psi(g^{-1}x)$$

determines a unitary representation of G on $L^2_\mu(G/H)$. Furthermore, the PVM A is covariant under the action of V^α:

$$V^\alpha(g)A(\Delta)V^\alpha(g)^{-1} = A(g\Delta)$$

[It follows that G acts as an automorphism of the closure of the span of the $A(\Delta)$.]

Axiom 4. Every elementary physical system ("particle") is described by a G-irreducible subspace of $L^2_\mu(G/H)$ for some symplectic homogeneous space G/H of G.

Let $\mathscr{H}$ denote a closed subspace of $L^2_\mu(G/H)$ hosting an irreducible representation and let $P: L^2_\mu(G/H) \to \mathscr{H}$ be the projection on $\mathscr{H}$. Then

$$\Delta \mapsto A_*(\Delta) \equiv PA(\Delta)P$$

$\Delta \in \text{Borel}(G/H)$, defines a covariant positive operator valued measure (POVM) on $\mathscr{H}$, also denoted "the phase space localization operator" (restricted to $\mathscr{H}$). Because P does not commute with each $A(\Delta)$, one does not obtain in this manner a projection valued measure on $\mathscr{H}$.

Axiom 5. A measurement apparatus may isolate an elementary system. In this circumstance, one may associate a projection P with the measurement apparatus, with P given as above. Then, for any density operator ρ in $L^2_\mu(G/H)$, the probability that a measurement of state ρ using this apparatus will yield an outcome in $\Delta \in \text{Borel}(G/H)$ is given by the expected values $\mathfrak{E}(\rho, PA(\Delta)P) = \text{Tr}(\rho PA(\Delta)P) = \text{Tr}(\rho A_*(\Delta))$.

Suppose $\mathscr{H}$ is a Hilbert space that is not necessarily a closed subspace of $L^2_\mu(G/H)$ but $\mathscr{H}$ hosts a representation U of G. Let A_* be a covariant POVM defined on the Borel sets of G/H that is absolutely continuous with respect to the invariant measure μ; then a general result shows that A_* has an operator density $T\colon x \mapsto T(x)$ that is measurable and

$$A_*(\Delta) = \int_\Delta d\mu(x)\, T(x)$$

$T(x)$ is a positive operator and T satisfies the covariance condition

$$U(g)T(x)U(g)^{-1} = T(gx)$$

If U is irreducible, $T(x)$ is a one-dimensional projection (for each $x \in G/H$). Set $T(\text{identity class}) \equiv |\eta\rangle\langle\eta|$ for some $\eta \in \mathscr{H}$. Then we shall denote T by T_η, A_* by A_η, and $\mathscr{H}$ by $\mathscr{H}_\eta$. For $K_\eta(x, y) \equiv \langle U(\sigma(x))\eta, U(\sigma(y))\eta\rangle$, $x, y \in G/H$, it follows that K_η defines a reproducing kernel in $L^2_\mu(G/H)$ whenever the representation U satisfies the technical condition of "square integrability over (G/H)." The K_η plays an important role in a rigorous treatment of quantum field theory (Schroeck, 1988). The existence of representations square-integrable over G/H for a general group G is not yet completely understood mathematically. When this does hold, the map

$$W_\eta\colon\quad \mathscr{H} \to L^2_\mu(G/H), \quad [W_\eta\psi](x) = \langle U(\sigma(x))\eta, \psi\rangle_{\mathscr{H}}$$

provides a canonical intertwining operator such that, for all $g \in G$,

$$W_\eta U(g) = V^\alpha(g)W_\eta$$

whenever η satisfies $U(h)\eta = \alpha(h)\eta$, $h \in W$. W_η is an isometry when η is suitably normalized. In this way, the representation space $\mathscr{H}$ may be identified with a subspace of $L^2_\mu(G/H)$, and for $P_\eta\colon L^2_\mu(G/H) \to W_\eta\mathscr{H}$, we have $W_\eta A_\eta(\Delta)W_\eta^{-1} = P_\eta A(\Delta)P_\eta$.

The mysterious "square-integrability over G/H" condition on η above is a condition of finite total (kinetic) energy in the cases $G = \text{Heisenberg}$, Galilei, or Poincaré group. For the Poincaré case, especially for massless particles, the form of W_η and the intertwining relation are, in fact, slightly more complicated than presented here, although they are close in form. *All* irreducible representations for these groups may be so entwined, that is, found as irreducible representations of $L^2_\mu(G/H)$ for some H (Brooke and Schroeck, 1993; Healy and Schroeck, 1993). Therefore, all of standard quantum theory may be found within this formalism. In the Poincaré case, the subgroup H and G/H for the massive case differ from those for the massless case.

From $A_\eta(\Delta) = \int_\Delta d_\mu(x)\, T_\eta(x)$ it follows that $A_\eta(\Delta)$ has purely discrete spectrum in [0, 1] for Δ compact (Schroeck, 1989). Then $\mathrm{Tr}(A_\eta(\Delta)) = \mu(\Delta)$. To localize a particle with wave vector $\psi \in \mathscr{H}_\eta$ in phase space volume Δ, one applies $A_\eta(\Delta)$; that is, $\psi \mapsto A_\eta(\Delta)\psi$. If ψ is an eigenvector of $A_\eta(\Delta)$ with eigenvalue λ, then $|A_\eta(\Delta)\psi(x)|^2 = \lambda^2|\psi(x)|^2$; so the process of localizing ψ in Δ reduces the probability of finding the particle in any region by the factor λ^2. λ^2 is called an attenuation factor. One computes that if Δ is a rectangle of sides ΔP and ΔQ, then $\mu(\Delta) = \Delta P \cdot \Delta Q \cdot (2\pi\hbar)^{-1}$. Thus if $\mu(\Delta) < 1$, that is, $\Delta P \cdot \Delta Q < 2\pi\hbar$, then no eigenvalue of $A_\eta(\Delta)$ is near 1 and the process of localization in Δ causes severe attenuation. This is a form of the uncertainty principle. If $\mu\Delta \gg 1$ and the boundary of Δ is smooth with normal to the boundary not twisting much, a result of Omnès (1989) shows that most eigenvalues of $A_\eta(\Delta)$ are either near 1 or near 0. Thus the number of eigenvalues near 1 is approximately $\mu(\Delta)$; that is, the number of independent eigenvectors not attenuated by localization is approximately $\mu(\Delta)$. This is a statement of quantization of space. In signal processing the same Heisenberg group analysis applies, and this result is known as "the channel capacity theorem."

2. ADVANTAGES OF THE FORMALISM

The advantages of the formalism of quantum mechanics on phase space arise from the property of informational completeness of the phase space localization operators:

Definition. A subset $\{A_\alpha | \alpha \in I\}$ of $\mathfrak{A}$ is informationally complete iff for $\rho, \rho' \in \mathfrak{S}$, $\mathfrak{E}(\rho, A_\alpha) = \mathfrak{E}(\rho', A_\alpha)$ $\forall \alpha \in I$ implies $\rho = \rho'$.

As an example, let E^P, E^Q be the spectral projections for momentum and position, respectively, in $L^2(\mathbb{R}^n)$ for ordinary spin-zero quantum theory. Then,

$$\{E^P(\Delta) | \Delta \in \mathrm{Borel}(\mathbb{R}^n)\} \cup \{E^Q(\Delta) | \Delta \in \mathrm{Borel}(\mathbb{R}^n)\}$$

is *not* an informationally complete set. The electrical engineering equivalent of this for $n = 1$ is that time-series analysis of signals plus frequency spectrum analysis of the same signal is not informationally complete.

Theorem (Healy and Schroeck, 1993). For $G =$ affine, Heisenberg, or Galilei group, and η square-integrable over G/H with $U(h)\eta = \alpha(h)\eta$ for all $h \in H$, then for spin-zero representations, $G/H \cong \mathbb{R}^n$ and $\{A_\eta(\Delta) | \Delta \in \text{Borel}(G/H)\}$ is informationally complete in $\mathscr{H}$. [A similar result holds for spin-j representations, but one needs a family $(2j+1)$-fold larger, in order to handle each spin component.]

An immediate consequence of this theorem coupled with a theorem of Busch (1991) is that every bounded operator, hence every observable, is contained in the closure of the span of $\{A_\eta(\Delta) | \Delta \in \text{Borel}(G/H)\}$. Thus, beginning with G, H, and η, we construct $L^2_\mu(G/H)$, the $A_\eta(\Delta)$, and finally all bounded operators, all observables. In this way, the fact that G is a group of symmetries for $\mathfrak{A}$ is automatic (Schroeck, 1989).

This example and theorem expose a crucial difference between joint phase space analysis of experiments and a common quantum analysis of experiments based on measurement of position alone, or position alone augmented by measurement of momentum alone. The latter types of experiment are incapable of being informationally complete. On the other hand, consider

$$\begin{aligned}
\mathfrak{E}(\rho, A_\eta(\Delta)) &= \text{Tr}(\rho A_\eta(\Delta)) \\
&= \text{Tr}\left(\rho \int_\Delta d\mu(x)\, T_\eta(x)\right) \\
&= \int_\Delta d\mu(x)\, \text{Tr}(\rho T_\eta(x)) \\
&= \int_\Delta d\mu(x)\, \langle U(\sigma(x))\eta, \rho U(\sigma(x))\eta \rangle
\end{aligned}$$

In the case that ρ is a vector state, $\rho = |\psi\rangle\langle\psi|$, then

$$\text{Tr}(\rho, A_\eta(\Delta)) = \int_\Delta d\mu(x)\, |\langle U(\sigma(x))\eta, \psi \rangle|^2$$

By the theorem above, the set of these numbers for $\Delta \in \text{Borel}(\mathbb{R}^n)$ is informationally complete. From this and basic integration theory, it is equivalent to know the set of numbers $\{|\langle U(\sigma(x))\eta, \psi\rangle|^2, x \in G/H\}$. Formulas showing how to reconstruct ψ up to a constant phase from the real numbers (transition probabilities) $|\langle U(\sigma(x))\eta, \psi\rangle|^2$ are known; similarly, formulas for reconstructing ρ from the set of numbers $\mathrm{Tr}(\rho T_\eta(x))$ are known (Healy and Schroeck, 1993).

We next show that these numbers are experimentally accessible in a practical way. These phase-space-dependent transition probabilities are directly observable in experiments in which particles are captured on a momentum- or frequency-dependent screen, as in color photographs, holographs, cloud chambers, etc. Alternatively, interferometry experiments may also yield the desired data; this is the content of the following theorem, which is a direct consequence of the superposition principle, i.e., of the existence of interference effects.

Theorem. Let η be a fixed vector in $L^2(\mathbb{R})$ with narrow momentum spread and finite energy. Let ψ be any vector in $L^2(\mathbb{R})$. Suppose there is a process whereby ψ and $U(g)\eta$, $g \in$ Heisenberg group, may be superimposed and the intensities

$$I(g) \equiv \|\psi + U(g)\eta\|^2$$

measured. Then the visibility (contrast) of the resulting interference pattern determines $|\langle U(g)\eta, \psi\rangle|$. The result generalizes to $L^2(\mathbb{R}^n)$.

Proof. Recall that for an interference pattern as in Fig. 1, the visibility $V(g_0)$ is defined by $V(g_0) \equiv [I_{\max} - I_{\min}]/[I_{\max} + I_{\min}]$, where $I_{\max}$ and $I_{\min}$ are measured near g_0. For $g \in$ Heisenberg group, write $g \equiv (q, p)$, q standing for a translation by q of position and p standing for a translation by p of momentum. Let p_0 denote the mean momentum of η. Then,

$$\langle U(q,p)\eta, \psi\rangle \cong \exp\{iq \cdot (p_0 - p) + i\theta(q,p)\}\, |\langle U(q,p)\eta, \psi\rangle|$$

where both $\theta(q,p)$ and $|\langle U(q,p)\eta, \psi\rangle|$ are slowly varying functions of (q,p). Thus,

$$\begin{aligned} I(q,p) &= \|\psi\|^2 + \|\eta\|^2 + 2\,\mathrm{Re}\langle U(q,p)\eta, \psi\rangle \\ &= \|\psi\|^2 + \|\eta\|^2 + 2\cos\{q \cdot (p_0 - p) + \theta(q,p)\}\, |\langle U(q,p)\eta, \psi\rangle| \end{aligned}$$

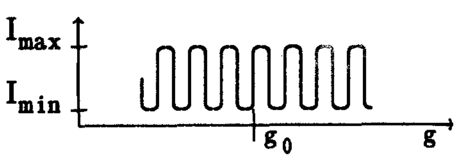


Fig. 1

Therefore, $I_{\max} - I_{\min} \cong 4|\langle U(q,p)\eta, \psi\rangle|$ and $I_{\max} + I_{\min} \cong 2(\|\psi\|^2 + \|\eta\|^2)$; so,

$$V(q,p) \cong c|\langle U(q,p)\eta, \psi\rangle|, \qquad c^{-1} = (\|\psi\|^2 + \|\eta\|^2)/2$$

The normalization of the $A_\eta(\Delta)$ will eliminate c; however, we note that if ψ and η are normalized, then $c = 1$. ■

A similar analysis holds if one can instead prepare a test particle of well-defined position. Note that delta functions are not required; realistic data analysis is possible.

Thus, in any experimental situation in which a test particle (test signal) of well-defined momentum (frequency) or position may be used, superposition allows the taking of informationally complete data.

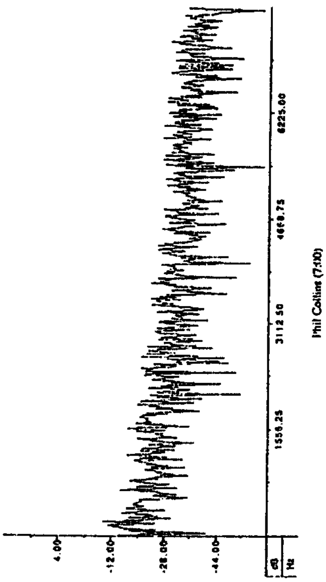
In contrast, one never measures the complex data $\langle U(q,p)\eta, \psi\rangle$ nor the complex signal "ψ." One can only measure in such a way as to obtain real readouts or energy patterns. All measurements of complex signals in the end boil down to such measurements, and it is efficient to deal directly with these measurement outcomes rather than converting with effort back to a complex wave interpretation of the data. We add that the terms $\langle U(g)\eta, \psi\rangle$ define the coherent state transform in general. Unfortunately, these terms are not easily experimentally accessible, in contrast to the terms $|\langle U(g)\eta, \psi\rangle|$.

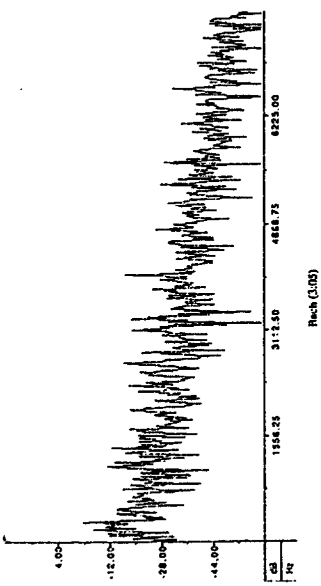
Consider the following:

Advantage List 1. Measurement of the phase space location operators to obtain $\mathrm{Tr}(\rho T_\eta(x))$ yields informationally complete data. In particular, this yields:

a. Efficient pattern recognition procedures.
b. Unique characterization of fixed noise, as well as practical subtraction of such noise from the desired signal.
c. Adaptive experimental processing (by varying the choice of η).
d. Data compression techniques.
e. Extension of the realm of applicability of quantum theory to general signal processing in both the electrical and biological realms.

To understand that (a)–(e) are advantages, consider that separate position (time) and momentum (frequency) analysis (Fourier analysis) possesses none of these properties! For example, the first line of the tune "Mary Had a Little Lamb" repeated periodically over and over with the same energy given to each note cannot be distinguished from a tune derived from it by permuting the words (notes) when using time-series and frequency-spectrum analysis. The time series and the frequency spectrum are


4.00
-12.00
-28.00
-44.00
dB
Hz
1556.25
3112.50
6225.00
Phil Collins (7:00)


4.00
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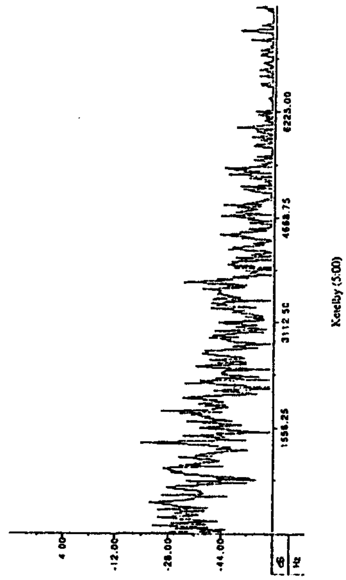
-28.00
-44.00
dB
Hz
1556.25
4668.75
6225.00
Bach (3:05)


4.00
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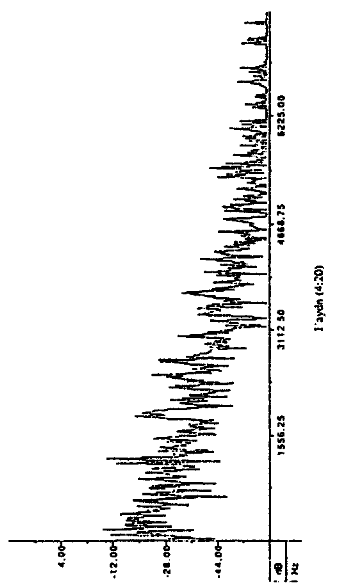
1556.25
4668.75
6225.00


4.00
-12.00
-28.00
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dB
Hz
1556.25
3112.50
4668.75
6225.00

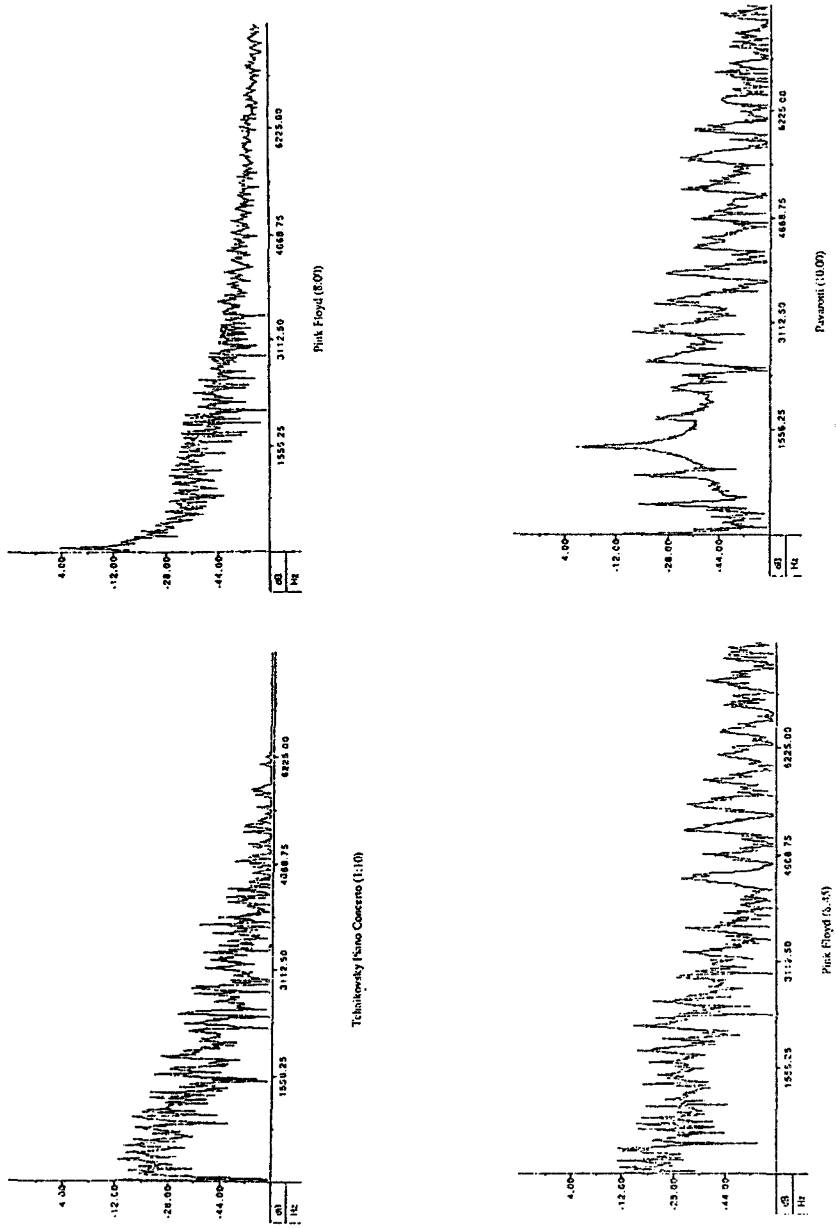
Pink Floyd (8:01)
Tchaikowsky Piano Concerto (1:10)
Pavarotti (10:00)
dB
Hz

Fig. 2

both invariant under permutations. In contrast, joint time–frequency analysis will uniquely determine the tune. To illustrate this, I have had a video prepared, *Fourier Transform: The Movie*, which compares these two methods of signal analysis (Ertem, 1992). The use of frequency spectra there shows the difficulty as well as the partial success of distinguishing very different sounds via Fourier analysis. On the other hand, a previously unrecognized characteristic of the voice of a famous Italian singer is revealed by joint time–frequency analysis. Snapshots of power spectra (energy versus frequency) from this video are presented in Fig. 2 for the purpose of distinguishing the sounds from pieces by Tchaikovsky, Bach, Haydn, Ketelby, Phil Collins, the rock group Pink Floyd, and Luciano Pavarotti. These spectra show the unsuitability of Fourier methods to distinguish these radically different sounds. There should be no surprise in this, since we are already familiar with the fact that marginal probability distributions do not determine a unique joint probability distribution. Here we are precisely dealing with joint quantum probabilities for nonindependent observables, and the following fundamental question arises: why should we persist in describing Hilbert space quantum observables in terms of expected values of position or of momentum or of their spectral families when observations may be more naturally described in terms of informationally complete POV measures on phase space? Moreover, why describe states in terms of complex-valued functions of position or of momentum rather than functions on phase space when the description of experiment is more suited to the latter?

For claim (b), consider the familiar weather radar screen showing noise from nearby buildings appearing as a large signal near the center of the radar screen. For radar systems operating as time-series systems or through Fourier transform as frequency spectra, subtraction of the noise signal as a function of time (or frequency) from the corresponding total signal tends to degrade the signal so much, it is not done. Signal analysis based on energy versus time and frequency jointly, being informationally complete, does not suffer from this. This newer type of analysis was suggested for implementation in radar analysis several decades back (Stutt and Spafford, 1968; Delong and Hofstetter, 1967, 1969).

For claim (c), one need only acknowledge the feasibility of using different states $|\eta\rangle\langle\eta|$ for test particles/test signals. Wave generators and particles in different environments or of different preparation may be used. Bats use this in echolocation in order to extract different characteristics of potential prey (Simmons *et al.*, 1975; Suga, 1990: Schroeck, 1991). No such flexibility exists in Fourier transform theory since there is only one Fourier transform.

For claim (d), one first observes that the transformation on density operators ρ given by

$$\rho \mapsto R_\eta(x) = \mathrm{Tr}(\rho T_\eta(x)), \qquad x \in G/H$$

yields a positive measurable function on G/H with integral = 1. That is, the density for the phase space localization operator yields a bona fide classical distribution function on the classical phase space G/H. One may compute the classical entropy for $R_\eta(x)$. For a set of choices of η, choose the one yielding lowest uncertainty to obtain a data compression process.

Even without the adaptive part of the procedure, simply using the general techniques for the Heisenberg group with (worst) test function η = Gaussian yields an enormous data compression in photo analysis (Daugman, 1988).

Claim (e) has already been justified in the discussion above. We do not wish to leave the impression that experiments more typical of physics lie outside the realm of application here. In fact, the results of Foulis and Schroeck (1990) show that the present formalism is a natural consequence of axiomatizing a Hilbert space description of physics from the operational manual point of view. Then the process of calibration of the instrument leads, by use of the Sakai operator in C^*-algebra theory, directly to the covariant POVM defined over a classical phase space. In a more applied paper (Busch and Schroeck, 1989), the POVM analysis of several basic experiments of physics is carried out.

Having introduced the dequantization map $\rho \mapsto R_\eta$ from quantum density operators to classical probability densities, we search for a corresponding quantization procedure from classical observables to quantum observables. In view of the informational completeness property for the phase space localization operators in the spin-zero case, one knows from the theorem of Busch (1991) that almost any observable B may be written in the form

$$B = \int_{G/H} f(x) T_\eta(x)\, d\mu(x)$$

for some measurable function f. Theorems of Healy and Schroeck (1993) allow one to determine f from B. In the other direction, any real-valued measurable function in $L^p_\mu(G/H)$ for any $1 \leq p \leq \infty$ yields a bounded operator which is even a compact operator if $p < \infty$ (Schroeck, 1989). Then

$$\begin{aligned}\mathrm{Tr}(\rho B) &= \int_{G/H} f(x)\, \mathrm{Tr}(\rho T_\eta(x))\, d\mu(x) \\ &= \int_{G/H} f(x) R_\eta(x)\, d\mu(x)\end{aligned}$$

that is, classical and quantum expectations agree. It is emphasized that no limit as $\hbar \to 0$ is taken. The quantum-to-classical correspondence is a direct consequence of the two properties (a) G/H is a symplectic space interpreted as a classical phase space, (b) the set of localization operators $\{A_\eta(\Delta) | \Delta \in \text{Borel}(G/H)\}$ is informationally complete. One may further check that the correspondence between commutator brackets for the quantum operators and Poisson brackets for the corresponding classical observables holds whenever one of the observables is in the Lie algebra for G (Schroeck, 1985). General commutations do not coincide because of the fact that the $A(\Delta)$ form only a POVM rather than a PVM. Since this is a direct consequence of the choice of instrument (and hence, of η) used to isolate the elementary physical system, this seems to be an essential ingredient rather than a drawback.

Advantage List 2:

f. A classical–quantum correspondence is present without introducing approximations or unphysical limits.

g. There is no need to use a classical description for the measurement process and a quantum description for the object to be measured. The two systems may be treated on an equal footing.

As a direct consequence of advantage (g), the question of objectification in classical measurement would seem to be equivalent to objectification in quantum measurement.

There is no particular reason to isolate elementary systems in all physical situations. In particular, one could choose to remain in the context of the reducible space $L^2_\mu(G/H)$. In this setting, quantum statistical mechanics and classical statistical mechanics may be given a unified treatment (Ali and Prugovečki, 1977).

Furthermore, the group G may be taken to be the Poincaré group or even more general relativistic groups such as the de Sitter group. In this fashion, once one obtains the reproducing kernel K_η, quantum fields may be rigorously defined to obtain quantum fields *on curved space* without singular behavior of the fields (Schroeck, 1988). Also, without introduction of quantum fields, one may obtain the phase space localization operators for Poincaré relativistic particles, including those of zero mass (Brooke and Schroeck, 1993). By comparison, localization of the photon in the formalism of standard quantum theory is well known to be impossible (Newton and Wigner, 1949; Wightman, 1962).

Finally, since every irreducible representation of the Heisenberg, Galilei, and Poincaré groups is contained in $L^2_\mu(G/H)$ for some H, every result of ordinary quantum theory is contained in quantum theory on phase space. Sometimes it is even easier to express the physics in this phase space context, as happens with the description of signals as energy as a function of phase space rather than as a complex wave over configuration space or as a complex wave on momentum space.

Advantage List 3:

h. Classical and quantum statistical mechanics may be treated with a common formalism.
i. Phase space localization of relativistic particles including those of zero mass (e.g., photon) is carried out.
j. Quantum field theory of free particles may be developed even in the setting of curved space.
k. All results of ordinary quantum theory are contained in the theory of quantum mechanics on phase space. The phase space description may even simplify the physical description.

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REFERENCES

Ali, S. T., and Prugovečki, E. (1977). Classical and quantum statistical mechanics in a common Liouville space, *Physica*, **89A**, 501–521.

Brooke, J. A., and Schroeck, F. E., Jr. (1993). Localization of the photon in phase space, preprint.

Busch, P. (1991). A characterization of informationally complete sets of physical quantities, *International Journal of Theoretical Physics*, **30**, 1217–1227.

Busch, P., and Schroeck, F. E., Jr. (1989). On the reality of spin and helicity, *Foundations of Physics*, **19**, 807–872.

Daugman, J. G. (1988). Relaxation neural network for non-orthogonal image transforms, *Proceedings IEEE International Conference on Neural Networks*, **1988**, I547–I560.

DeLong, D. F., and Hofstetter, E. M. (1967). On the design of optimum radar waveforms for clutter rejection, *IEEE Transactions on Information Theory*, **IT-13**, 454–463.

DeLong, D. F., and Hofstetter, E. M. (1969). The design of clutter-resistant radar waveforms with limited dynamic range, *IEEE Transactions on Information Theory*, **IT-15**, 376–385.

Ertem, C. (1992). *Fourier Transform: The Movie*, Ertem Enterprises, Boca Raton, Florida.

Foulis, D., and Schroeck, F. E., Jr. (1990). Stochastic quantum mechanics viewed from the language of manuals, *Foundations of Physics*, **20**, 823–858.

Healy, D. M., and Schroeck, F. E., Jr. (1993). On informational completeness of covariant observables and Wigner coefficients, preprint, Dartmouth College.

Newton, T. D., and Wigner, E. P. (1949). Localized states for elementary systems, *Reviews of Modern Physics*, **21**, 400–406.

Omnès, R. (1989). Logical reformulation of quantum mechanics. IV. Projectors in semiclassical physics, *Journal of Statistical Physics*, **57**, 357–382.

Schroeck, F. E., Jr. (1985). The dequantization programme for stochastic quantum mechanics, *Journal of Mathematical Physics*, **26**, 306–310.
Schroeck, F. E., Jr. (1988). Quantum fields on reproducing kernel Hilbert spaces, *Reports on Mathematical Physics*, **26**, 197–209.
Schroeck, F. E., Jr. (1989). Coexistence of observables, *International Journal of Theoretical Physics*, **28**, 247–262.
Schroeck, F. E., Jr. (1991). Unsharpness in measurement yields informational completeness: Implementations in quantum signal processing and natural biological systems, in *Proceedings of the Symposium on the Foundations of Modern Physics 1990*, P. Lahti and P. Mittelstaedt, eds., World Scientific, Singapore, pp. 375–389.
Simmons, J. A., Howell, D. J., and Suga, N. (1975). Information content of bat sonar echoes, *American Scientist*, **63**, 204–215.
Stutt, C. A., and Spafford, L. J. (1968). A 'best' mismatched filter response for radar clutter discrimination, *IEEE Transactions on Information Theory*, **IT-14**, 280–287.
Suga, N. (1990). Biosonar and neural computation in bats, *Scientific American*, **1990**(June), 60–68.
Wightman, A. S. (1992). On the localizability of quantum mechanical systems, *Review of Modern Physics*, **34**, 845–872.